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THE OBSERVABILITY OF LINEAR SINGULARLY PERTURBED SYSTEMS IN STATE SPACE†

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The state-space method is used to investigate the complete x- and y-relative observability of linear stationary singularly perturbed (LSSP) systems. Criteria, necessary conditions and sufficient conditions, phrased in terms of matrix ranks, are obtained for the observability; they involve the solutions of the defining equations, which are recurrent algebraic matrix equations. Duality principles are established between the LSSP observed and the control systems with coefficients of varying scales, the LSSP observed systems and LSSP control systems, and the governing equations for the observed and control systems. An example is given.

1. STATEMENT OF THE PROBLEM. DEFINITIONS

SUPPOSE the behaviour of a moving object is described by a linear stationary singularly perturbed (LSSP) system of differential equations

$$\begin{gathered}\dot{x}(t)=A_1x(t)+A_2y(t)\\ \mu\dot{y}(t)=A_3x(t)+A_4y(t),\quad t\geqslant t_0\\ x\in R^{n_1},\quad y\in R^{n_2},\quad 0<\mu\leqslant\mu^0\ll 1\end{gathered}\tag{1.1}$$

where μ is a small positive parameter and A_i $(i=1, 2, 3, 4)$ are constant matrices of the appropriate orders. Physically speaking, μ represents all the small parameters for which the dimensions of the state space Ω of system (1.1) are n_1+n_2: $\Omega\subset R^{n_1+n_2}$, $\Omega\triangleq\{\mathrm{col}(x, y)$: $x\in R^{n_1}$, $y\in R^{n_2}\}$. When $\mu=0$ the variable y is no longer a state vector and the dimensions of system (1.1) decrease to n_1.

We will assume that as a result of the current initial state $\{x(t_0), y(t_0)\}$ and parameters $\mu\in(0, \mu^0]$ a transient $x(t, \mu), y(t, \mu)$ of system (1.1) has begun. Let us also assume that neither the initial state $\{x(t_0), y(t_0)\}$ nor the trajectory $\{x(t, \mu), y(t, \mu)\}$ are accessible to direct measurement. The observer, over a time interval $T=[t_0, t_1]$, can measure the output vector-valued function $w(t)$ of a measuring device, which is governed by the rule

$$\begin{gathered}w(t)=D_1x(t,\mu)+D_2y(t,\mu),\quad t\in T\\ \mu\in(0,\mu^0],\quad w\in R^{n_3},\quad n_3\leqslant n_1+n_2\end{gathered}\tag{1.2}$$

where, if $n_3\leqslant n_1+n_2$ the $(n_1+n_2)\times(n_1+n_2)$ matrix $\|D_1, D_2\|$ is singular.

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The observed system (1.1) with output function (1.2) will be referred to as the LSSP observed system (LSSPOS).

The problem of complete (x-, y-*relative) observability.* Given an LSSPOS (1.1), (1.2), it is required to find conditions under which, given measurements $w(t)$, $t \in T$, one can uniquely reconstruct the initial state $\{x(t_0), y(t_0)\}$ (the component $x(t_0)$, and the component $y(t_0)$ of the initial state $\{x(t_0), y(t_0)\}$) of system (1.1) that, for a given $\mu \in (0, \mu_0]$ produces the given output (1.2).

Definition 1. The LSSP system (1.1) is completely (x-, y-relatively) observable by output (1.2) in $T=[t_0, t_1]$ if its problem of complete (x-, y-relative) observability is solvable for any initial state $\{x(t_0), y(t_0)\} \in R^{n_1+n_2}$, $\mu \in (0, \mu^0]$.

The aim of this paper is to find the conditions for the observability of a LSSPOS (1.1), (1.2), in terms of the matrices A_i $(i=1,2,3,4)$, D_j $(j=1,2)$ of the system.

2. THE GOVERNING EQUATIONS OF THE OBSERVED SYSTEM

To formulate the conditions for the observability of system (1.1), (1.2), we define $n_1 \times (n_1+n_2)$ matrices X_k^i, $n_2 \times (n_1+n_2)$ matrices Y_k^i, and $n_3 \times (n_1+n_2)$ matrices W_k^i $(i, k = 0, 1, 2, \ldots)$ as follows:

$$\begin{aligned} &X_k^i \triangleq \| X_{kj}^i,\ j=1,2\|, \quad X_{k1}^i \in R^{n_1 \times n_1}, \quad X_{k2}^i \in R^{n_1 \times n_2} \\ &Y_k^i \triangleq \| Y_{kj}^i,\ j=1,2\|, \quad Y_{k1}^i \in R^{n_2 \times n_1}, \quad Y_{k2}^i \in R^{n_2 \times n_2} \\ &W_k^i \triangleq \| W_{kj}^i,\ j=1,2\|, \quad W_{k1}^i \in R^{n_3 \times n_1}, \quad W_{k2}^i \in R^{n_3 \times n_2} \end{aligned} \tag{2.1}$$

The presence of the subscript j $(j=1, 2)$ in the matrices X_{kj}^i, Y_{kj}^i, W_{kj}^i is due to the different scales of magnitude of the variables x, y in system (1.1), (1.2), indicating the existence of motions with two quite different velocities $\dot{x}(t)$, $\dot{y}(t)$.

We establish a correspondence between the vector-valued functions $x(t)$, $y(t)$, $w(t)$ and matrices X_k^i, Y_k^i, W_k^i, stipulating that

$$\begin{aligned} &x \to X_k^i, \quad \dot{x} \to X_{k+1}^i \\ &y \to Y_k^i, \quad \mu\dot{y} \to Y_{k+1}^{i+1}, \quad w \to W_k^i \end{aligned} \tag{2.2}$$

where the subscript $k+j$ $(j=0, 1)$ in the matrices X_{k+j}^{i+l}, Y_{k+j}^{i+l}, W_{k+j}^{i+l} represents the jth derivative of the vectors x, y, w and the superscript $i+l$ $(l=0, 1)$ represents the lth degree of the factor μ multiplying the derivatives $\dot{x}(t)$, $\dot{y}(t)$. Then by (2.2) the system of differential equations (1.1) is transformed into an algebraic system of matrix equations, which are recurrent in i, k

$$\begin{aligned} &X_{k+1}^i = A_1 X_k^i + A_2 Y_k^i \\ &Y_{k+1}^{i+1} = A_3 X_k^i + A_4 Y_k^i, \quad k=0,1,2,\ldots, \quad i=0,1,2,\ldots, \quad k-1 \end{aligned} \tag{2.3}$$

and the output corresponds to an algebraic matrix equation, also recurrent in i, k

$$W_k^i = D_1 X_k^i + D_2 Y_k^i, \quad k=0,1,2,\ldots, \quad i=0,1,2,\ldots, \quad k-1 \tag{2.4}$$

To ensure that (2.3) and (2.4) have a unique solution, we set

$$X_0^0 = \| E_{n_1}, 0_{n_1 \times n_2} \|, \quad X_k^i = 0_{n_1 \times (n_1+n_2)}, \quad i > k, \quad k = 0,1,2,\ldots; \quad i < 0 \vee k < 0$$
$$Y_0^0 = \| 0_{n_2 \times n_1}, E_{n_2} \|, \quad Y_k^i = 0_{n_2 \times (n_1+n_2)}, \quad i \geq k+1, \quad k = 0,1,2,\ldots; \quad i \leq 0 \vee k < 0 \tag{2.5}$$

By analogy with [1], the recurrent equations (2.3) and (2.4) will be referred to as the defining equations of the LSSPOS (1.1), (1.2), and the matrices X_{kj}^i, Y_{kj}^i, W_{kj}^i $(k=0, 1, 2, \ldots; i=0, 1, 2, \ldots, k-1; i=0, 1, 2)$ computed from (2.3), (2.4) with initial conditions (2.5) will be referred to as the components of solutions $\{X_k^i, Y_k^i, W_k^i\}$ $(k=0, 1, 2, \ldots, i=0, 1, 2, \ldots, k-1)$ of the governing equations of system (2.3)–(2.5).

By introducing the governing equations (2.3) and (2.4) we have made it possible to investigate, instead of the differential system (1.1) with output (1.2), a system of algebraic matrix equations, thus obtaining effective conditions for observability of the LSSPOS (1.1), (1.2) expressed in terms of the parameters A_i $(i=1, 2, 3, 4)$, D_j $(j=1, 2)$ of the system.

The governing equations may be derived in another form (albeit related to (2.3)), which yields different algebraic conditions for observability. These new equations have certain advantages over (2.3) and (2.4).

We express the LSSPOS (1.1) and (1.2) in the state space $\Omega \subset R^{n_1+n_2}$ as a system

$$\dot{z}(t) = A(\mu)z(t), \quad z \in R^{n_1+n_2} \tag{2.6}$$

$$w(t) = Dz(t), \quad w \in R^{n_3}, \quad t \in T, \quad \mu \in (0, \mu^0] \tag{2.7}$$

which depends, singularly as $\mu \to 0$, on the parameter μ. In these equations

$$D \triangleq \| D_1, D_2 \|, \quad z \in \Omega, \quad z(t) \triangleq \left\| \begin{matrix} x(t) \\ y(t) \end{matrix} \right\|, \quad A(\mu) \triangleq \left\| \begin{matrix} A_1 & A_2 \\ A_3/\mu & A_4/\mu \end{matrix} \right\| \tag{2.8}$$

Define matrices $H_x \triangleq \| E_{n_1}, 0_{n_1 \times n_2} \|$, $H_y \triangleq \| 0_{n_2 \times n_1}, E_{n_2} \|$. Then, obviously, the problem of the complete (x-, y-relative) observability of system (1.1), (1.2) is equivalent to the problem of the complete (H_x-, H_y-relative) observability for system (2.6) by output (2.7). By analogy with [1], the governing equation for system (2.6), (2.7) for any $\mu \in (0, \mu_0]$ is

$$Z_{k+1} = A(\mu)Z_k, \quad Z_0 = E_{n_1+n_2}, \quad k = 0,1,2,\ldots \tag{2.9}$$

$$W_k = DZ_k, \quad Z_k \in R^{(n_1+n_2) \times (n_1+n_2)}, \quad W_k \in R^{n_3 \times (n_1+n_2)} \tag{2.10}$$

Lemma 1. The solutions Z_k of the governing equation (2.9), for each k $(k=0, 1, 2, \ldots)$ are related to the solutions X_k^i, Y_k^i of the governing equations (2.3) as follows:

$$Z_k = \left\| \begin{matrix} \sum\limits_{m=0}^{k} \mu^{m-k} X_k^{k-m} \\ \sum\limits_{m=0}^{k} \mu^{m-k} Y_k^{k-m} \end{matrix} \right\|, \quad k = 0,1,2,\ldots \tag{2.11}$$

The proof proceeds by induction, relying on the equalities $X_{l+1}^{l+1} = 0$, $Y_{l+1}^0 = 0$, which follow from (2.5).

Lemma 2. For every k $(k=0, 1, 2, \ldots)$ the matrices W_k of (2.10) and W_k^i of (2.4) are related as follows:

$$W_k = \sum_{m=0}^{k} \mu^{m-k} W_k^{k-m} \tag{2.12}$$

The proof is by induction on k, relying on formulae (2.10), (2.4) and Lemma 1.

3. ADJOINT CONTROL SYSTEMS AND THEIR GOVERNING EQUATIONS

Together with the observed system (2.6), (2.7), with its singular dependence on μ as $\mu \to 0$, let us consider the adjoint control system

$$\dot{z}(t) = -A'(\mu)z(t) + D'u(t), \quad t \in T \tag{3.1}$$

which, by (2.8), may be written as

$$\begin{aligned} \dot{x}(t) &= -A_1'x(t) - (A_3'/\mu)y(t) + D_1'u(t) \\ \dot{y}(t) &= -A_2'x(t) - (A_4'/\mu)y(t) + D_2'u(t), \quad t \in T \end{aligned} \tag{3.2}$$

and is a control system with differently scaled coefficients: "small" coefficients $-A_1'$, $-A_2'$ and "large" ones $-A_3'/\mu$, $-A_4'/\mu$. Clearly, the control system (3.2) is adjoint to the observed system (1.1), (1.2).

Defining matrices $X_k^{(c)i} \in R^{n_1 \times n_3}$, $Y_k^{(c)i} \in R^{n_2 \times n_3}$ $(i,\ k = 0,\ 1,\ 2,\ \ldots)$, we set up a correspondence between the vector-valued functions $x(t), y(t)$, and the matrices $X_k^{(c)i}$, $Y_k^{(c)i}$, governed by the rule

$$\begin{aligned} &x \to X_k^{(c)i}, \quad \dot{x} \to X_{k+1}^{(c)i} \\ &\mu^{-1}y \to Y_k^{(c)i-1}, \quad \dot{y} \to Y_{k+1}^{(c)i} \end{aligned} \tag{3.3}$$

where the subscript $k+j$ $(j = 0,\ 1)$ in the matrices $X_{k+j}^{(c)i+l}$, $Y_{k+j}^{(c)i+l}$ corresponds to the jth derivative of $x(t), y(t)$, and the superscript $i+l$ $(l = -1,\ 0)$ to the lth degree of the factor μ multiplying the variables $x(t)$ and $y(t)$ (compare with (2.2)). Then, by (3.3), the system of differential equations (3.2) determines a system of i, k-recurrent algebraic matrix equations

$$\begin{aligned} X_{k+1}^{(c)i} &= -A_1'X_k^{(c)i} - A_3'Y_k^{(c)i-1} \\ Y_{k+1}^{(c)i} &= -A_2'X_k^{(c)i} - A_4'Y_k^{(c)i-1}, \quad k = 0,1,2,\ldots, \quad i = 0,1,2,\ldots,k \end{aligned} \tag{3.4}$$

which we shall solve for initial data

$$\begin{aligned} X_0^{(c)0} &= D_1', \quad X_k^{(c)i} = 0_{n_1 \times n_3}, \quad i < 0 \vee k < 0 \vee i < k \\ Y_0^{(c)0} &= D_2', \quad Y_k^{(c)i} := 0_{n_2 \times r_3}, \quad i < 0 \vee k < 0 \vee i > k \end{aligned} \tag{3.5}$$

To establish the relationship between Eqs (2.3), (3.4) and their solutions, we define, for every quadruple of indices i, j, k, l, matrices of order $(n_1 + n_2) \times (n_1 + n_2)$

$$Z_{kl}^{ij} \triangleq \begin{Vmatrix} X_k^i \\ Y_l^j \end{Vmatrix}, \quad Z_{kl}^{ij} = \| z_{kln}^{ij}, \quad n = 0,1,2,\ldots,n_1 + n_2 - 1 \|$$

where X_k^i, Y_l^j are the components of the solutions of the governing equations (2.3), (2.5), and matrices of order $(n_1 + n_2) \times n_3$

$$Z_{kl}^{(c)ij} \triangleq \begin{Vmatrix} X_k^{(c)-i} \\ Y_l^{(c)i} \end{Vmatrix}, \quad Z_{kl}^{(c)ij} = \| z_{kln}^{(c)ij}, \quad n = 0,1,2,\ldots,n_1 + n_2 - 1 \|$$

where $X_k^{(c)i}$, $Y_l^{(c)j}$ are the components of the solutions of the governing equations (3.4), (3.5), and z_{kln}^{ij}, $z_{klm}^{(c)ij}$ are the columns of the matrices Z_{kl}^{ij}, $Z_{kl}^{(c)ij}$, respectively. Form the set

$$Z = \{z_{kln}^{ij}, z_{klm}^{(c)ij},\ n = 0,1,2,\ldots,\ n_1+n_2-1;\ m = 0,1,2,\ldots,\ n_3-1;\ i,j,k,l = 0,1,2,\ldots\}$$

$$Z \subset R^{(n_1+n_2)\times\infty}$$

and define a linear operator $L\colon\ Z \to Z$, $L = E_{n_1+n_2} - A(\mu)\exp(-p_{jkl})$, where $p_{jkl} \triangleq \partial^3/\partial_j\partial_k\partial_l$ is the differentiation operator, $\exp(-p_{jkl})$ a shift operator defined on the indices: $\exp(-p_{jkl})z_{kl}^{ij} = z_{k-1,l-1}^{i,j-1}$. Then the operator adjoint to L, $L^*\colon\ Z \to Z$, may be written as

$$L^* = E_{n_1+n_2} + A'(\mu)\exp(-p_{jkl}) \tag{3.6}$$

The governing equations (2.3) may obviously be written in operator notation as follows:

$$L(Z_{k+1,k+1}^{i,i+1}) = 0_{(n_1+n_2)\times(n_1+n_2)} \tag{3.7}$$

and the governing equations (3.4) as

$$L^*(Z_{k+1,k+1}^{(c)i,i}) = 0_{(n_1+n_2)\times n_3} \tag{3.8}$$

Comparing Eqs (3.7) and (3.8), we see that system (2.3) is the adjoint to system (3.4), and we shall accordingly refer to the recurrent equations (3.4) as the adjoint governing equations of system (1.1), (1.2), and to $X_k^{(c)i}$, $Y_k^{(c)i}$ $(i,\ k = 0,\ 1,\ 2,\ \ldots)$ evaluated by (3.4), (3.5), as the components of the adjoint governing equations (3.4), (3.5).

4. THE OBSERVABILITY OF LINEAR TIME-INDEPENDENT SINGULARLY PERTURBED SYSTEMS

We will now formulate the observability conditions for the LSSPOS (1.1), (1.2) in terms of the components of the solutions of the governing equations (2.3)–(2.5). To do this we consider the following matrices of order $n_3(n_1+n_2)\times(n_1+n_2)$

$$Q^z(\mu) \triangleq \left\| \begin{matrix} W_k \\ k = 0,1,2,\ldots,n_1+n_2-1 \end{matrix} \right\|,\quad Q(\mu) \triangleq \left\| \begin{matrix} \sum\limits_{m=0}^{k} \mu^m W_k^{k-m} \\ k = 0,1,2,\ldots,n_1+n_2-1 \end{matrix} \right\| \tag{4.1}$$

whose elements are the components of the solutions of the governing equations (2.9), (2.10) and (2.3)–(2.5), respectively. We shall call $Q(\mu)$ the observability matrix of system (1.1), (1.2).

Lemma 3. The matrices $Q^z(\mu)$ and $Q(\mu)$ have equal ranks

$$\operatorname{rank} Q^z(\mu) = \operatorname{rank} Q(\mu),\quad \mu \in (0, \mu^0]$$

The proof, which is obvious, follows from the form (4.1) of the matrices $Q^z(\mu)$, $Q(\mu)$ and from Lemma 2.

Theorem 1. An LSSP system (1.1) is completely observable by output (1.2) if and only if the observability matrix $Q(\mu)$ is of maximum rank

$$\operatorname{rank} Q(\mu) = n_1 + n_2,\quad \mu \in (0, \mu^0] \tag{4.2}$$

The proof follows from the representation of system (1.1), (1.2) in the form (2.6), (2.7); confining our attention to $\mu \in (0,\ \mu^0]$, one then uses the criterion [1] $\operatorname{rank} Q^z(\mu) = n_1 + n_2$ for the complete observability of system (2.6), (2.7), and Lemma 3.

Remarks. 1. If system (1.1), (1.2) is completely observable for some $\mu^* \in (0, \mu^0]$, a number $\mu_* \leq \mu^*$ obviously exists such that (1.1) and (1.2) are completely observable for all $\mu \in (\mu_*, \mu^*]$.

2. Obviously, if (4.2) is true for all $\mu \in (0, \mu^0]$, then system (1.1) is completely observable by output (1.2) for all $\mu \in (0, \mu^0]$.

Using a result in [2] concerning the relative observability of system (2.6), (2.7) for $\mu \in (0, \mu^0]$ we can state the following corollary.

Corollary 1. The LSSPOS (1.1), (1.2) is x-relatively (y-relatively) observable if and only if

$$\operatorname{rank} Q(\mu) = \operatorname{rank}\begin{Vmatrix} H_x \\ Q(\mu) \end{Vmatrix}, \quad H_x = \| E_{n_1}, 0_{n_1 \times n_2} \|, \quad \mu \in (0, \mu^0]$$

$$\left(\operatorname{rank} Q(\mu) = \operatorname{rank}\begin{Vmatrix} H_y \\ Q(\mu) \end{Vmatrix}, \quad H_y = \| 0_{n_2 \times n_1}, E_{n_2} \| \right)$$

To formulate the next theorem we use a lemma, whose complete proof may be found elsewhere.†

Lemma 4. suppose we are given $n \times l$ matrices M_i $(i = 0, 1, 2, \ldots, n \leq l)$ and a number k $(k = 0, 1, 2, \ldots)$. If there exists m $(0 \leq m \leq k)$ for which rank $M_m = n$, then there exists $\mu_k > 0$ such that, for all $\mu \in (0, \mu_k]$

$$\operatorname{rank} \sum_{i=0}^{k} \mu^i M_i = n$$

Define matrices P of order $n_3(n_1 + n_2) \times (n_1 + n_2)$, $Q_1(\mu)$ of order $n_3(n_1 + n_2) \times n_1$, and $Q_2(\mu)$ of order $n_3(n_1 + n_2) \times n_2$, as follows:

$$P \triangleq \begin{Vmatrix} W_{k1}^{k-m+l_1} & W_{k2}^{k-m+l_2} \\ k = 0, 1, 2, \ldots, & n_1 + n_2 - 1 \end{Vmatrix}, \quad Q_1(\mu) \triangleq \begin{Vmatrix} \sum_{m=0}^{k} \mu^m W_{k1}^{k-m} \\ k = 0, 1, 2, \ldots, n_1 + n_2 - 1 \end{Vmatrix}$$

$$Q_2(\mu) \triangleq \begin{Vmatrix} \sum_{m=0}^{k} \mu^m W_{k2}^{k-m} \\ k = 0, 1, 2, \ldots, n_1 + n_2 - 1 \end{Vmatrix}$$

Using Theorem 1, Corollary 1 and the fact that matrix rank is preserved under multiplication of rows and columns by a non-zero number, as well as Lemma 4, we can now state sufficient conditions for the observability of system (1.1), (1.2), without referring to the parameter μ.

Theorem 2. Suppose that for some l_1, l_2 $(l_i = 0, 1, 2, \ldots, n_1 + n_2 - 1,\ i = 1, 2)$ there exists m $(m = 0, 1, 2, \ldots, n_1 + n_2 + \max(l_1, l_2) - 1)$ such that

$$\operatorname{rank} P = n_1 + n_2$$

$$\left(\operatorname{rank} P = \operatorname{rank}\begin{Vmatrix} H_x \\ P \end{Vmatrix}, \quad \operatorname{rank} P = \operatorname{rank}\begin{Vmatrix} H_y \\ P \end{Vmatrix} \right)$$

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Then there exists $\mu^* > 0$ such that the LSSPOS (1.1), (1.2) is completely (x-, y-relatively) observable for all $\mu \in (0, \mu^*]$.

Theorem 1 and Corollary 1 imply the following theorem.

Theorem 3. A necessary condition for the LSSPOS (1.1), (1.2) to be completely (x- and y-relatively) observable is that

$$\operatorname{rank} Q_1(\mu) = n_1, \quad \operatorname{rank} Q_2(\mu) = n_2, \quad \mu \in (0, \mu^0]$$
$$(\operatorname{rank} Q_1(\mu) = n_1, \quad \operatorname{rank} Q_2(\mu) = n_2)$$

In order to check the given system (1.1), (1.2) for observability in practice, it is more economical to use other conditions, requiring less computer resources. To that end we will derive observability conditions for the system in terms of solutions of the governing equations (3.4) and (3.5).

Define a matrix of order $n_3(n_1 + n_2) \times (n_1 + n_2)$

$$Q^{(c)}(\mu) \triangleq \left\| \begin{matrix} \sum_{m=0}^{k} \mu^m (X_k^{(c)k-m})', & \sum_{m=0}^{k} \mu^m (Y_k^{(c)k-m})' \\ k = 0, 1, 2, \ldots, n_1 + n_2 - 1 & \end{matrix} \right\| \tag{4.3}$$

where $X_k^{(c)i}$, $Y_k^{(c)i}$ are the solutions of the governing equations (3.4) and (3.5). We will call $Q^{(c)}(\mu)$, like the matrix $Q(\mu)$ of (4.1), the observability matrix of system (1.1), (1.2), since $Q^{(c)}(\mu)$ is the transpose of the observability matrix $(Q^{(c)}(\mu))'$ of system (3.2), which is the adjoint of system (1.1), (1.2).

Theorem 4. The LSSP system (1.1) is complete (x-, y-relatively) observable by output (1.2) if and only if

$$\operatorname{rank} Q^{(c)}(\mu) = n_1 + n_2, \quad \mu \in (0, \mu^0] \tag{4.4}$$

The proof, which is omitted here, is analogous to that of Theorem 1, using the relative observability criterion established in [2] for systems (2.6) and (2.7) for $\mu \in (0, \mu^0]$, as well as formulae (2.8), (3.4), (3.5) and (4.3).

The necessary and sufficient conditions for the observability of system (1.1), (1.2), expressed in terms of solutions of the governing equations (3.4) and (3.5) are literal repetitions of Theorems 2 and 3, with $\| W_{k1}^i, W_{k2}^i \|$ replaced by $\| (X_k^{(c)i}, Y_k^{(c)i})' \|$, respectively.

The reader will observe that verification of the observability conditions (4.2), which are formulated in terms of the components W_k^i of the solutions of the governing equations (2.3), (2.4) and (2.5), requires working out Eq. (2.3) $n_1 + n_2 - 1$ times and Eq. (2.4) $n_1 + n_2$ times; this requires a total of

$$\left[\frac{1 + (n_1 + n_2)}{2} (n_1 + n_2 + n_3) - 1 \right] (n_1 + n_2)^2 (2n_1 + 2n_2 - 1)$$

multiplication and addition operations. To verify conditions (4.4) of Theorem 4 in terms of the solutions $X_k^{(c)i}$, $Y_k^{(c)i}$ of the governing equations (3.4) and (3.5), it is necessary to work out the two equations of (3.4) $n_1 + n_2$ times, that is, to carry out

$$\left[\frac{1 + (n_1 + n_2)}{2} (n_1 + n_2) - 1 \right] n_3 (n_1 + n_2)(2n_1 + 2n_2 - 1)$$

multiplication and addition operations. Since by assumption $n_3 \leqslant n_1 + n_2$, when n_1 and n_2 are large this procedure considerably reduces the demands on computer time and memory in a computerized analysis of LSSPOS such as (1.1) and (1.2) for observability.

5. THE DUALITY OF SINGULAR OBSERVED AND CONTROL SYSTEMS

Together with the LSSPOS (1.1), (1.2), let us consider the adjoint system of equations (3.2) with differently scaled coefficients and initial data

$$x(t_0)=x_0,\quad y(t_0)=y_0,\quad x_0\in R^{n_1},\quad y_0\in R^{n_2} \tag{5.1}$$

Suppose that over the interval $T=[t_0,\ t_1]$ we have a class $U(t)$ of continuous n_3-vector functions $u(t)$, henceforth called admissible controls: $u(t)\in U(t)$.

The complete (x-, y-*relative) controllability problem.* Determine the conditions under which, for any (n_1+n_2)-vectors $(x_0,\ y_0)$, $(x_1,\ y_1)$ and $\mu\in(0,\ \mu^0]$, an admissible control $u(t)$ exists such that the corresponding solution $[x(t,\ \mu),\ y(t,\ \mu)]$ $t\in T$ (component $x(t,\ \mu)$ and component $y(t,\mu)$ of the solution), of system (3.2), (5.1) will satisfy the condition $x(t_1,\ \mu)=x_1,\ y(t_1,\ \mu)=y_1$ $(x(t_1,\ \mu)=x_1,\ y(t_1,\ \mu)=y_1)$.

Definition 2. System (3.2), (5.1) is completely (x-, y-relatively) controllable in T if its controllability problem has a solution for any $(x_0,\ y_0)\in R^{n_1+n_2}$, $(x_1,\ y_1)\in R^{n_1+n_2}$, $\mu\in(0,\ \mu^0]$.

Theorem 5 (*first duality principle*). The LSSPOS (1.1) and (1.2) is completely controllable if and only if the linear time-independent system with differently scaled coefficients (3.2), (5.1) is completely controllable in T.

The proof follows directly from the representation of the observed system (1.1), (1.2) in the form (2.6), (2.7), the representation of the control system (3.2) in the form (3.1), and the usual duality principle [3] for the resulting linear time-independent systems for $\mu\in(0,\ \mu^0]$.

An obvious corollary of Theorems 1, 4 and 5 is the following.

Corollary 2. If

$$\operatorname{rank}\left\| \begin{matrix} \sum\limits_{m=0}^{k}\mu^m(W_{k1}^{k-m})' \\ \\ \sum\limits_{m=0}^{k}\mu^m(W_{k2}^{k-m})' \end{matrix} \quad k=0,1,2,\ldots,n_1+n_2-1 \right\| = n_1+n_2,\quad \mu\in(0,\mu^0]$$

or

$$\operatorname{rank}\left\| \begin{matrix} \sum\limits_{m=0}^{k}\mu^m X_k^{(c)k-m} \\ \\ \sum\limits_{m=0}^{k}\mu^m Y_k^{(c)k-m} \end{matrix} \quad k=0,1,2,\ldots,n_1+n_2-1 \right\| = n_1+n_2,\quad \mu\in(0,\mu^0]$$

then system (3.2), (5.1) is completely controllable in T.

There is another duality principle for the controllability and observability of LSSP systems. To establish it, we consider, together with any LSSPOS (1.1), (1.2), the LSSP control system (LSSPCS)

$$\begin{aligned} \dot{x}(t)&=-A_1'x(t)-A_3'y(t)+D_1'u(t),\quad x\in R^{n_1},\quad y\in R^{n_2}, \\ \mu\dot{y}(t)&=-A_2'x(t)-A_4'y(t)+D_2'u(t),\quad u\in R^{n_3},\quad t\in T,\quad \mu\in(0,\mu^0] \end{aligned} \tag{5.2}$$

with initial data (5.1). One can formulate the complete (x-, y-relative) controllability problem for the LSSPCS (5.2), (5.1), in the same way as for system (3.2), (5.1). To obtain conditions for the controllability of system (5.2), (5.1), we define matrices $X^i_{(c)k} \in R^{n_1 \times n_3}$, $Y^i_{(c)k} \in R^{n_2 \times n_3}$ $(i,\ k = 0, 1, 2, \ldots)$ and set up a correspondence between the vector functions $x(t)$, $y(t)$ and matrices $X^i_{(c)k}$, $Y^i_{(c)k}$ according to the rule (2.2). Then, by (2.2), the system of differential equations (5.2) will correspond to an algebraic system of (i, k)-recurrent matrix equations

$$\begin{aligned} X^i_{(c)k+1} &= -A_1' X^i_{(c)k} - A_3' Y^i_{(c)k} \\ Y^{i+1}_{(c)k+1} &= -A_2' X^i_{(c)k} - A_4' Y^i_{(c)k}, \quad i, k = 0, 1, 2, \ldots \end{aligned} \tag{5.3}$$

which we shall solve with initial data

$$\begin{aligned} X^0_{(c)0} &= D_1', \quad X^i_{(c)k} = 0_{n_1 \times n_3}, \quad i > k \vee (i < 0) \vee (k < 0) \\ Y^0_{(c)0} &= D_2', \quad Y^i_{(c)k} = 0_{n_2 \times n_3}, \quad i > (k+1) \vee (i \leqslant 0) \vee (k = 0) \end{aligned} \tag{5.4}$$

We shall call Eqs (5.3) the governing equations of the LSSPCS (5.2) and (5.1), while the matrices $X^i_{(c)k}$, $Y^i_{(c)k}$ $(i,\ k = 0, 1, 2, \ldots)$ given by (5.3) and (5.4) will be called solutions of the governing equations (5.3) and (5.1). Since system (5.3) may be written in terms of the adjoint operator (3.6)

$$L^*(Z^{i,i+1}_{(c)k+1,k+1}) = 0_{(n_1+n_2) \times n_3}, \quad Z^{ij}_{(c)kl} \triangleq \begin{Vmatrix} X^i_{(c)k} \\ Y^j_{(c)l} \end{Vmatrix}$$

it follows that the governing equations (5.3) of the LSSPCS (5.2) are the adjoints of the governing equations (2.3) of the LSSPOS (1.1).

The relationship between the solutions $X^i_{(c)k}$, $Y^i_{(c)k}$ of (5.3) and the solutions $X^{(c)i}_k$, $Y^{(c)i}_k$ of (3.4) is established by the following lemma.

Lemma 5. For every i, k $(i,\ k = 0, 1, 2, \ldots)$ the solutions $X^i_{c(k)}$ of the governing equations (5.3), (5.4) and the solutions $X^{(c)i}_k$, $Y^{(c)i}_k$ of the governing equations (3.4), (3.5) satisfy the relations $X^i_{(c)k} = X^{(c)i}_k$, $Y^{i+1}_{(c)k} = Y^{(c)i}_k$.

The proof is obtained by comparing the governing equations (5.3) and (3.4).

Theorem 6 (*second duality principle*). The LSSPOS (1.1) and (1.2) is completely observable if and only if the LSSPCS (5.2) and (5.1) is completely controllable in T.

Proof. By a criterion proved in [4]

$$\operatorname{rank} \begin{Vmatrix} \sum_{m=0}^{k} \mu^m X^{k-m}_{(c)k} & \\ & k = 0, 1, 2, \ldots, n_1 + n_2 - 1 \\ \sum_{m=0}^{k} \mu^m Y^{k-m+1}_{(c)k} & \end{Vmatrix} = n_1 + n_2, \quad \mu \in (0, \mu^0] \tag{5.5}$$

for the completely controllability of the LSSPCS (5.2), Lemma 5 and the fact that the ranks of a matrix and its transpose are equal, it follows that condition (5.5) is equivalent to condition (4.4) of Theorem 4. The theorem is proved.

Corollary 3. A linear time-independent system with differently scaled coefficients (3.2) is completely controllable in T if and only if the LSSPCS (5.2) has that property.

In other words, systems (3.2) and (5.2) are equivalent in the sense of controllability.

6. EXAMPLE

The following is a linear stationary singularly perturbed model of the rotation of an elastic link in an electromechanical manipulatory robot [5]

$$\dot{x}(t)=y(t),\ \ \mu\dot{y}(t)=-y(t)+u(t),\ \ \mu\in(0,\mu^0],\ \ x\in R,\ y\in R,\ u\in R \tag{6.1}$$

Let us consider the observation of this system by measuring the output function

$$w(t)=ax(t)+by(t),\ \ w\in R,\ \ t\in T=[t_0,t_1] \tag{6.2}$$

where a and b are parameters and u is a given control, $\|u\|\leqslant 1$.

Problem. Determine for what values of the parameters a and b the LSSP system (6.1) is completely observable by output (6.2).

Express system (6.1) in the same form as (1.1), where $n_1=n_2=n_3=1$, $A_1=0$, $A_2=1$, $A_3=0$, $A_4=-1$, $B_1=0$, $B_2=1$, $D_1=a$, $D_2=b$. Using the governing equations in the form (2.3), we have

$$X_0^0=\|1,0\|,\ \ X_1^0=\|0,1\|,\ \ X_1^1=\|0,0\|$$

$$Y_0^0=\|0,1\|,\ \ Y_1^0=\|0,0\|,\ \ Y_1^1=\|0,-1\|$$

$$W_0^0=\|a,b\|,\ \ W_1^0=\|0,a\|,\ \ W_1^1=\|0,-b\|$$

Then the observability matrix $Q(\mu)$ of (4.1) for system (6.1), (6.2) is

$$Q(\mu)=\begin{Vmatrix} W_0^0 \\ W_1^1+\mu W_1^0 \end{Vmatrix}=\begin{Vmatrix} a & b \\ 0 & -b+\mu a \end{Vmatrix}$$

whence it follows by Theorem 1 that for $a\neq 0$, $b\neq 0$, and all $\mu\in(0,\ b/a-\varepsilon)$, where ε is any number, $\varepsilon\ll 1$, or for $a\neq 0$, $b=0$ and any $\mu\in(0,\ \mu^0]$, we have rank $Q(\mu)=2$, i.e. system (6.1), (6.2) is completely observable in T. For $a=0$, $b\neq 0$, the system is not completely observable, but for $l_1=0$, $l_2=1$, $m=0$ it follows from Theorem 2 that it is y-relatively observable for all $\mu\in(0,\ \mu^0]$, since in that case $W_{01}^0=0$, $W_{02}^1=0$, $W_{11}^0=0$, $W_{12}^1=-b$ and

$$\operatorname{rank}\begin{Vmatrix} 0 & 0 \\ 0 & -b \end{Vmatrix}=\operatorname{rank}\begin{Vmatrix} 0 & 1 \\ 0 & 0 \\ 0 & -b \end{Vmatrix}=1$$

7. CONCLUSIONS

We have proposed that the observability of the LSSP system (1.1), (1.2) be investigated by the state-space method. This method, which goes back to Kalman [3], requires neither the construction of an asymptotic expansion of the solutions of system (1.1) in powers of the small parameter μ [6], nor the traditional condition det $A_4\neq 0$. The method treats the problem globally (in terms of μ) and yields an effective criterion for the observability of system (1.1), (1.2), expressed in terms of solutions of the defining equations of the system. The latter constitute a system of recurrent algebraic matrix equations, derived in accordance with an explicit rule from the original observed system (1.1), (1.2). The singularity of system (1.1) enables one to formulate sufficient conditions for observability (Theorem 2) that do not involve the small parameter μ.

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